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Some Spaces in Which Best Uniform Approximation is Never Possible

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Let X be a compact Hausdorff space and A a Banach subspace of $C_R(X)$ in the uniform norm, with unit. Let $f \in C_R(X)$ such that $0 < d = \inf_{g \in A} \| f - g \|$. Let μ be a regular Borel measure on X such that $\| \mu \| = 1$, $\mu \perp A$ and $\mu(f) = d$. Let $M(A, f)$ be the set of all such measures.

Remark. If $\langle \operatorname{supp} \mu \rangle$ is connected for some $\mu \in M(A, f)$, then f has no best approximation in A.

Proof. Let g be a best approximation. Then $| g - f | = d$ on $\langle \operatorname{supp} \mu \rangle$, so $g - f = d$ (or $-d$, we assume d) on $\langle \operatorname{supp} \mu \rangle$. Thus

$$d = \int f \, d\mu = \int (f - g) \, d\mu = \int - d \, d\mu = 0,$$

a contradiction which establishes the result.

We proceed now to find compact plane sets, X, and Banach spaces $A \subset B \subset C_R(X)$ such that best uniform approximations of elements of B by elements of A *never* exist.

We first collect some well known facts from the theory of rational approximation. Let X be a compact plane set such that the interior of X, X^0, is connected and $\langle X^0 \rangle = X$. Let $R(X)$ be the uniform closure on X of the space of rational functions with poles off X. Let K be the topological boundary of X. We assume $m(K) = 0$ where m is Lebesque area measure. Let ν be a regular Borel measure in $R(X)^{\perp}$. Set

$$\hat{\nu}(z) = \int \frac{1}{\zeta - z} \, d\nu(\zeta).$$

Then it is well known [1, Chapter 2] that $\nu = 0$ if and only if $\hat{\nu} = 0$ a.e. (m) and that $\hat{\nu} = 0$ on the complement, X^c, of X. Suppose that $K \neq \langle \operatorname{supp} \nu \rangle$. (Since the Shilov boundary of $R(X)$ is a subset of K, we may assume $\langle \operatorname{supp} \nu \rangle \subset K$.) Then there must be a relatively open subset of K and hence an open set U such that $U \cap \langle \operatorname{supp} \nu \rangle = \varnothing$, $U \cap X^0 \neq \varnothing$, and $U \cap X^c \neq \varnothing$. But $\hat{\nu}$ is clearly analytic off $\langle \operatorname{supp} \nu \rangle$ and hence must vanish on X^0. We summarize the above in the following lemma.

LEMMA. *Let X be a compact plane set such that X^0 is connected, $\langle X^0 \rangle = X$ and $m(K) = 0$. Then either $\nu = 0$ or $\langle \operatorname{supp} \nu \rangle = K$.*

Now let $D(X) = \{f \in C_R(X): f \text{ is harmonic on } X^0\}$, and let A be the uniform closure on X of $\{u: u = \operatorname{Re} f, f \in R(X)\}$.

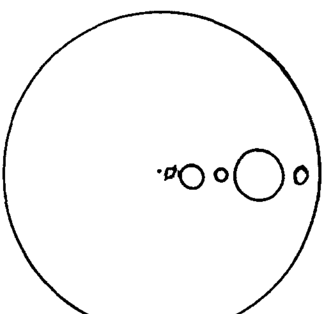
THEOREM. *Let X be a compact plane set such that X^0 is connected, $\langle X^0 \rangle = X$, $m(K) = 0$ and K is connected. Let $f \in D(X)$. If $f \notin A$, there is no best uniform approximation to f from A.*

Proof. Let ν be a real measure in $M(A, f)$. Then $\nu \perp R(X)$ so $K = \langle \operatorname{supp} \nu \rangle$. If $g \in A$ is a best approximation to f, then $g - f$ is constant on $\langle \operatorname{supp} \nu \rangle$ and hence on K. By the maximum modulus principle $g - f$ is constant on X and hence $f \in A$, a contradiction. Q.E.D.

It is not difficult to find compact plane sets satisfying the hypotheses of Theorem 2 and such that $D(X) \neq A$. For example, set

$$X = \langle \Delta \rangle (0, 1) \Big\backslash \bigcup_{i=1}^{\infty} \Delta(a_i, r_i),$$

where $0 < a_i < 1$, $r_i < \frac{1}{2}$, $1 \leqslant i < \infty$, and $\langle \Delta \rangle (a_i, r_i) \cap \langle \Delta \rangle (a_j, r_j) = \varnothing$ for $i \neq j$. By choosing a_i and r_i appropriately we can insure that the set of nonregular (in the sense of Wiener) points on $K \cap [0, 1]$ has positive length,



and such that K is connected. Then $A \neq D(X)$ [2, p. 31].

We are especially interested in the following situation: Let

$$A(X) = \{f \in C(X): f \text{ is analytic on } X^0\}.$$

Conjecture. If $f \in A(X)$ and $f \notin R(X)$ then f has no best uniform approximation in $R(X)$.

If best uniform approximation exists it is unique. The conjecture is somewhat in contrast to the result of Hintzman in [3, pp. 1062–1066].

References

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